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# Improving Volatility Forecasts Using Market-elicited Ambiguity Aversion Information

Raymond H.Y. So<sup>†</sup> and Tarik Driouchi<sup>‡</sup>

## Abstract

Distinguishing between risk and uncertainty, this paper proposes a volatility forecasting framework that incorporates asymmetric ambiguity shocks in the (exponential) GARCH-M conditional volatility process. Spanning 25 years of daily data and considering the differential role of ambiguity attitudes in the gain and loss domains, our models capture a rich set of information and provide more accurate volatility forecasts both in-sample and out-of-sample when compared to ambiguity-free or risk-based counterparts. Volatility-timing trading strategies confirm the economic significance of our proposed framework and indicate that an annualized excess return of 3.2% over the benchmark could be earned from 1995 to 2014.

*Keywords:* Decision theory, Model uncertainty, Ambiguity, Volatility

*JEL Codes:* G41, D81, D91

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# Improving Volatility Forecasts Using Market-elicited Ambiguity Aversion Information

## **Abstract**

Distinguishing between risk and uncertainty, this paper proposes a volatility forecasting framework that incorporates asymmetric ambiguity shocks in the (exponential) GARCH-M conditional volatility process. Spanning 25 years of daily data and considering the differential role of ambiguity attitudes in the gain and loss domains, our models capture a rich set of information and provide more accurate volatility forecasts both in-sample and out-of-sample when compared to ambiguity-free or risk-based counterparts. Volatility-timing trading strategies confirm the economic significance of our proposed framework and indicate that an annualized excess return of 3.2% over the benchmark could be earned from 1995 to 2014.

# 1. INTRODUCTION

Pioneered by Engle (1982), the autoregressive conditional heteroskedasticity (ARCH) volatility modeling approach has revolutionized the way we predict volatility and allowed understanding time-varying volatility under different assumptions. Engle's seminal approach in modeling conditional volatility has been extended to the generalized ARCH (GARCH, Bollerslev 1986), exponential ARCH (E(G)ARCH, Nelson 1991), and GJR-(G)ARCH models (Glosten et al. 1993). In recent years, an important stream of research concerned with the notion of ambiguity, as uncertainty beyond probabilistic risk, has emerged to highlight the relevance and significance of model uncertainty in asset pricing, volatility prediction, policy evaluation, and decision making (e.g., Manski 2000; Cao et al. 2005; Brock et al. 2007; Agliardi and Agliardi 2009; Easley and O'Hara 2009; Kast et al. 2014). The concept of uncertainty and its distinction from risk was highlighted almost a century ago by Knight (1921), was further conceptualized by Keynes (1921, 1937), and corroborated by Ellsberg (1961) in his famous thought experiments on individual decision making under ambiguity. Although ambiguity has been widely recognized and theorized by researchers in the context of financial markets (see for e.g., Gilboa and Schmeidler 1989; Chateauneuf et al. 1996; Cao et al. 2005; Handel et al. 2013), empirical work linking model uncertainty to volatility modeling is still scarce (e.g., Buraschi and Jiltsov 2006; Anderson et al. 2009; Fan and Mancini 2009; Driouchi et al., 2016). This is due to the inherent difficulties in quantifying ambiguity empirically.

Motivated by the need to quantify ambiguity and assess the potential implications of model uncertainty in volatility prediction, this paper investigates the empirical relation between ambiguity attitudes and risk in financial markets and highlights the value of incorporating ambiguity in GARCH volatility forecasts. In a recent paper, Driouchi et al. (2016) study the lead-lag relationship between ambiguity implied by option prices and realized volatility around the subprime crisis in a standard historical variance setting, and demonstrate that forward-looking ambiguity can be important in volatility prediction especially in uncertain times (i.e., 2006-2008). In their effort to estimate the impact of uncertainty on expected returns, Anderson et al. (2009) also examined the effect of uncertainty on conditional volatility as a robustness check. However,

their study only focuses on quarterly data, as limited by the availability of professional forecasters' survey data, and does not provide information on how uncertainty affects conditional volatility in higher frequency settings (e.g., daily). Also related, Fan and Mancini (2009) show how accounting for learning and model misspecification in option pricing can minimize empirical pricing errors and improve volatility prediction. They validate their approach using option pricing data for the 2002-2004 period. No study has highlighted the role of ambiguity aversion, as a behavioral construct, in volatility forecasting over an extensive time window in- and out-of-sample and using a large dataset of option prices in the context of GARCH volatility.

We fill this gap in research by examining the relationship between investors' attitudes to ambiguity, as inferred from market traded option prices, and conditional volatility over the 1990-2014 period. More specifically, we extract investors' attitudes to ambiguity from S&P 500 index options using a modified option pricing formula under ambiguity and account for ambiguity innovations in our GARCH volatility forecasts. This approach allows us to capture and quantify the ambiguity attitudes of sophisticated options investors/traders on a real-time basis. Our paper differs from that of Driouchi et al. (2016) in that we explicitly incorporate ambiguity innovations in the GARCH methodology, control for downside and upside markets (i.e., gains vs. losses), and assess economic significance and forecasting accuracy in- and out-of-sample over the entire 1990-2014 period. Not concerned with the GARCH apparatus, Driouchi et al. (2016) are focused on the subprime crisis and the incremental information content of ambiguity implied from put option prices over 2006-2008 in a standard historical variance setting. By allowing asymmetric<sup>1</sup> uncertainty shocks in the GARCH conditional volatility process for gains and losses, we show that option market ambiguity attitudes (OMAA) are quantitatively important in determining the subsequent level of conditional volatility. Our analysis reveals a strong relationship between OMAA and ex post conditional volatility over a quarter century of daily data. Ambiguity aversion is positively associated with ex post conditional volatility in the gain domain, while negatively associated with ex post conditional volatility in

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<sup>1</sup> The asymmetric property of volatility has been well documented. For example, Ewing and Mali (2010) estimate volatility persistence in GARCH models by introducing an asymmetric structural break variable. Tse (2016) finds that asymmetric volatility generates long-run skewness.

the loss domain. This special S-shape relationship has been prominent in the behavioral economics, decision theory and psychology literatures concerned with agents' decision making behavior. We unveil it in the context of financial markets and GARCH volatility forecasting. Our results are robust to a range of forecasting tests (e.g., in-sample, out-of-sample, superior predictive ability, and economic significance) and various modeling specifications.

Back in the 1990s, many studies suggested that the relationship between an individual's decision and her ambiguity attitude may not be explained by a simple linear relationship, especially when considering emotional sensitivities to gains and losses (Thaler et al. 1997; Tversky and Kahneman 1986; Thaler and Johnson 1990; Low 2004). For example, Viscusi and Chesson (1999) explained how an individual's ambiguity attitude may shift from ambiguity aversion to ambiguity seeking (and vice versa) under the fear and hope effects. Their work underlines the differential role of ambiguity attitudes in the gain and loss domains. They suggest that in the gain domain, subjects are more ambiguity averse for high probabilities of gains but become more ambiguity seeking for low probabilities of gains. On the other hand, in the loss domain, subjects are more ambiguity seeking for high probabilities of loss and more ambiguity averse for low probabilities of loss. A similar shift in ambiguity preference is also documented by Ho, Keller and Keltyka (2002) and Chkravarty and Roy (2008). Kelsey et al. (2011) also point out that under Knightian uncertainty or ambiguity investors react differently to past winners and losers, and explain how momentum profitability relates to such an asymmetry. In line with this pattern of attitudes, we posit that the different impacts of positive and negative returns on investors' ambiguity and other sentimental attributes might explain the difficulties researchers have faced in modeling the interaction between market risk and ambiguity empirically. This is especially important in our analysis of the relationship between volatility and ambiguity, as the sought-after association might change from time to time depending on the level of gain/loss on the investment.

Inspired by the observed pattern of shifting ambiguity attitudes in decision making experiments, this paper analyses the relationship between volatility and ambiguity by taking into account gains/losses of market investors, and examines the extent to which adopting an ambiguity-based approach to modeling

volatility can contribute to improving the accuracy and practical relevance of GARCH volatility forecasts. Given the behavioral observations of Viscusi and Chesson (1999) and that behavioral biases from ambiguity neutrality create instability, we expect ambiguity aversion (seeking) to contribute to upward revisions in conditional volatility in the gain (loss) domain.

Our proposed GARCH volatility modeling methodology incorporates the asymmetric impact of option market ambiguity attitudes on ex post conditional volatility and shows that ambiguity, as proxied by OMAA, can improve the forecasting accuracy of GARCH volatility models using daily data that spans 25 years from 1990 to 2014. The inclusion of OMAA yields significant improvements in the in-sample root mean squared error (RMSE) for up to 13.1% versus the standard benchmark. We also assess the out-of-sample forecasting ability of our ambiguity-based volatility models when compared to their unambiguous and risk-based benchmark counterparts. Analyses based on different estimation windows, forecasting windows, sampling frequencies of intraday realized volatility, and four different loss functions, consistently confirm that OMAA is statistically significant in improving the accuracy of both in-sample and out-of-sample volatility forecasts. As a robustness test, we also examine the economic significance of considering OMAA in GARCH volatility by comparing portfolio returns generated from two simple volatility timing trading strategies based on our out-of-sample volatility forecasts under ambiguity. For the out-of-sample estimation window 1995-2014, an annualized 3.2% return can be earned in excess of that generated by the unambiguous or risk-based forecasts. We contribute to the literature by presenting robust and extensive empirical evidence on the importance of investors' miscalibration and the efficiency of option market information in GARCH volatility forecasting.

The paper is organized as follows. Section 2 specifies our ambiguity-based volatility models. Section 3 describes the data and variables. Section 4 presents the in-sample estimation results, out-of-sample forecasting performance and economic significance of our ambiguity-based GARCH volatility models. Section 5 concludes.

## 2. EMPIRICAL FRAMEWORK

To investigate the role of ambiguity attitudes in the formation of subsequent conditional variance of excess returns, we employ the GARCH-in-mean model and account for ambiguity and option implied variance as exogenous variables in the variance equation. For robustness, we examine the same linkage under an exponential version of GARCH to ensure positivity of forecasted variance. We expand our study out-of-sample to verify if the relationship observed from the in-sample estimation is stable and consistent in out-of-sample settings, and whether the discovered relationship provides economic value to volatility forecasting practice.

### 2.1 *Inferring Ambiguity Attitudes from the Option Market*

To assess the relationship between investors' ambiguity attitudes and ex post conditional variance, daily estimations of ambiguity attitudes are crucial. To this aim, we employ the rank dependent utility framework under ambiguity proposed by Chateauneuf et al. (1996) and extended to option pricing by Driouchi et al. (2015). Underlying the expanded option pricing framework is a modified version of geometric Brownian motion that allows subjective attitudes towards Knightian uncertainty to come into play. Related to the uncertain expected utility of Gul and Pesendorfer (2014), this type of Brownian motion has been validated by Kast and Lapied (2010) and Kast et al. (2014) in a number of decision theoretic contributions. The ambiguity-based Brownian motion is, thus, specified as follows:

$$\frac{dS}{S} = (\mu + m\sigma)dt + s\sigma dz \quad (\forall m \in ]-1,1[, \forall s \in ]0,1]) \quad (1)$$

$$dW = mdt + sdz \quad (2)$$

where  $S$  is the price of the underlying asset,  $m$  and  $s$  are the mean and standard deviation of a general Wiener process  $W$ ,  $z$  is a standard Wiener process. Agent's model misspecification under ambiguity is summarized by a capacity variable  $c$  which determines  $m$  and  $s$ , where  $0 < c < 1$  (Kast et al. 2014). When agents are faced with uncertainty, they are unsure about which of many probability distributions are correct. They assign weights to probability occurrences forming subjective probability expectations. The resulting subjective probabilities may be non-linear and even non-additive. The above model captures non-additive



probability features and miscalibration through the Choquet capacity function. Driouchi et al. (2015) employ this set of Brownian motions in (1) to derive an ambiguity-adjusted model for European exchange option pricing under Knightian uncertainty. As a special case of that model, the price of a European call<sup>2</sup> option with fixed strike  $K$  under ambiguity takes the following modified or ambiguity-adjusted Black-Scholes form:<sup>3</sup>

$$P_t^C = S_0 e^{-\delta' T} N\left(\frac{\ln\left(\frac{S_0}{K}\right) + (r' - \delta' + 0.5(s\sigma)^2)T}{s\sigma\sqrt{T}}\right) - K e^{-r' T} N\left(\frac{\ln\left(\frac{S_0}{K}\right) - (r' - \delta' + 0.5(s\sigma)^2)T}{s\sigma\sqrt{T}}\right) \quad (3)$$

where:

$$r' = r + m \frac{[r - (\mu + m\sigma)]}{s^2 \sigma}; \quad (4)$$

$$\delta' = \delta - \frac{(m + s^2 \sigma - s\sigma)[(\mu + m\sigma) - r]}{s^2 \sigma}$$

$$m = 2c - 1 \text{ and } s = \sqrt{4c(1-c)} \quad (\forall c \in ]0, 1[) \quad (5)$$

where  $P_t^C$  is the ambiguity-adjusted option price for  $0 < c < 1$ ,  $K$  is the strike price of the option,  $r$  is the risk-free rate,  $T$  is the time to maturity in years,  $\mu$  is the required return,  $\sigma$  is the volatility measure,  $c$  is the capacity variable proxying for miscalibration,  $r'$  is the subjective discount rate and  $\delta'$  is the subjective dividend yield.

To obtain an estimate of investors' ambiguity attitudes, we invert Eq. (3) numerically by minimizing the absolute deviations between the model price and market price:

$$OMAA_t \equiv c_t^* = \arg \min_{c|0 < c < 1} [|P_t^C(S_t, K, r, T, \sigma_t, \mu_t, \delta_t, c_t) - P_t^{Mkt}|] \quad (6)$$

---

<sup>2</sup> As robustness, we also obtained OMAA using VIX-implied model prices with comparable results. The alternative procedure consists of 1) taking VIX as the implied volatility of at-the-money options, 2) converting implied volatility into option prices using the standard Black-Scholes model, and 3) extracting OMAA from the VIX-implied option prices using Eq. (3). Results are in line with the OMAA from traded option prices presented herein. The additional VIX-based OMAA results are available from the authors.

<sup>3</sup> Technical details of the model are included in the supplementary appendix.

where  $P_t^{Mkt}$  is the market traded SPX option price,  $S_t$  is the closing level of S&P 500 index on day  $t$ ,  $\sigma_t$  is the index volatility,  $c_t$  is the time-varying ambiguity measure,  $\mu_t$  is the rate of return on the index, and  $\delta_t$  is the dividend yield of S&P 500 portfolio. Investors' ambiguity attitudes are summarized by the capacity variable  $c_t$  through the minimization of the absolute error function in (6). The resulting capacity variable  $c_t$  from (6) is our proxy for option market ambiguity attitudes (OMAA). In the extraction process, we take the 1-month LIBOR rate as the risk-free rate, the trailing twelve months dividend yield of the S&P 500 index as dividend yield, RiskMetrics EWMA volatility<sup>4</sup> (JP Morgan 1996) as the volatility input ( $\sigma_t$ ), and 12-month historical returns as a proxy for the rate of return<sup>5</sup> ( $\mu_t$ ). The choice of inputs in the extraction process is in line with the relevant literature (Barberis, Huang and Santos 2001; Gonzalez-Riveria, Lee and Mishra 2004; Harris and Nguyen 2013).

## 2.2 *GARCH-in-mean Estimations*

To begin with our specifications of econometric model, we assume the following mean equation for all the ARCH class models used in this paper:

$$r_t = \alpha_0 + \alpha_1 h_t + \varepsilon_t \quad (7)$$

where  $r_t$  is the daily logarithmic return of the S&P 500 index in excess of the logarithmic yield of 3-month treasury bill,  $h_t$  is the conditional variance at time  $t$ , which is estimated dynamically in the variance equation, and  $\varepsilon_t$  represents the unexpected excess return at time  $t$ . Rearranging terms, we get:

$$\varepsilon_t = r_t - (\alpha_0 + \alpha_1 h_t) \quad (8)$$

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<sup>4</sup> We have also used out-of-sample GARCH(1,1) with a three-year rolling estimation window and a simple 22-day standard deviation of returns as alternative measures of volatility. Our results are not crucially affected by the choice of volatility measure.

<sup>5</sup> The choice of proxy for the rate of return relates to investors' memory about past returns. Barberis, Huang and Santos (2001) emphasize the importance of investors' memory in determining their required returns. They show that investors tend have a short memory when recalling past gains and losses. We employ the 12-month past returns as our proxy for the rate of return as this time frame matches the short memory assumption plus gives a reliable sample size of 252 trading days. As a robustness check, we also extracted OMAA using shorter (6 months) and longer (up to 3 years) periods of returns as proxies for  $\mu_t$  and found the conclusions still hold.

In order to compare and assess the information content of ambiguity attitudes in determining ex post conditional variance, we first estimate a benchmark vanilla GARCH model without investors' ambiguity attitudes innovations. We define the variance equation of the benchmark Model 1.1 as:

*Model 1.1:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} \quad (9)$$

To assess the information content of the variance model with ambiguity attitudes under gains and losses, we specify two variance Models 1.2 and 1.3 as follows:

*Model 1.2:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 D_{t-1|r_{t-1}>0} OMAA_{t-1} \quad (10)$$

*Model 1.3:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 (1 - D_{t-1|r_{t-1}>0}) OMAA_{t-1} \quad (11)$$

where  $D_{t-1|r_{t-1}>0}$  is a dummy variable that takes a value of 1 when the excess return is positive and 0 otherwise, and  $OMAA_{t-1}$  is the ambiguity attitude measure. The dummy variables in Models 1.2 and 1.3 aim to capture the asymmetric effects of ambiguity attitude and ex post conditional variance in the gain (Model 1.2) and loss (Model 1.3) domains in line with ambiguity theory predictions. Since the inferred ambiguity attitude is based on option market information and to ensure the additional information content provided by ambiguity innovation is not due to informational overlaps with implied variance (despite the low correlation between OMAA and IV), we specify three additional models with option implied variance<sup>6</sup> as one of the exogenous variables:

*Model 1.4:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 IV_{t-1} \quad (12)$$

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<sup>6</sup> We have also investigated the role of volatility of volatility (as measured by CBOE VVIX index) in volatility modeling with a reduced sample period from 2007 to 2014 (since VVIX data is only available from 2007). In general VVIX is not significant under both GARCH-M and EGARCH-M estimations, while OMAA remains significant in the presence of VVIX.

where  $IV_{t-1}$  is the daily implied variance of the S&P 500 index. Model 1.4 represents the vanilla GARCH specification with implied variance and without the ambiguity attitudes innovation. To differentiate between gains and losses, we again consider the following two models with ambiguity:

*Model 1.5:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 IV_{t-1} + \beta_4 D_{t-1|r_{t-1}>0} OMAA_{t-1} \quad (13)$$

*Model 1.6:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 IV_{t-1} + \beta_4 (1 - D_{t-1|r_{t-1}>0}) OMAA_{t-1} \quad (14)$$

In addition to implied variance, we also control for the variance risk premium (VRP), a measure recently established to be a proxy for risk aversion (Bollerslev, Tauchen and Zhou 2009; Bollerslev, Gibson and Zhou 2011; Bekaert and Hoerova 2014), to ensure the incremental information content from ambiguity attitudes is not due to any information overlap with risk aversion. We add the daily variance risk premium to Models 1.4-1.6 yielding the following three specifications:

*Model 1.7:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 IV_{t-1} + \beta_4 VRP_{t-1} \quad (15)$$

*Model 1.8:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 IV_{t-1} + \beta_4 VRP_{t-1} + \beta_5 D_{t-1|r_{t-1}>0} OMAA_{t-1} \quad (16)$$

*Model 1.9:*

$$h_t = \beta_0 + \beta_1 \varepsilon_{t-1}^2 + \beta_2 h_{t-1} + \beta_3 IV_{t-1} + \beta_4 VRP_{t-1} + \beta_5 (1 - D_{t-1|r_{t-1}>0}) OMAA_{t-1} \quad (17)$$

### 2.3 EGARCH-in-mean Estimations

The introduction of exogenous variables in ARCH class models may sometimes suffer from yielding negative conditional variances. As a robustness test and to ensure positivity in our out-of-sample volatility forecasts, we follow Engle (1982) and Nelson (1991), and consider equivalents of Models 1.1 to 1.9 in exponential form:

Model 2.1:

$$\log(h_t) = \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) \quad (18)$$

Model 2.2:

$$\log(h_t) = \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 D_{t-1|r_{t-1}>0} OMAA_{t-1} \quad (19)$$

Model 2.3:

$$\log(h_t) = \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 (1 - D_{t-1|r_{t-1}>0}) OMAA_{t-1} \quad (20)$$

Model 2.4:

$$\log(h_t) = \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 \log(IV_{t-1}) \quad (21)$$

Model 2.5:

$$\log(h_t) = \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 \log(IV_{t-1}) + \beta_4 D_{t-1|r_{t-1}>0} OMAA_{t-1} \quad (22)$$

Model 2.6:

$$\begin{aligned} \log(h_t) = & \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 \log(IV_{t-1}) \\ & + \beta_4 (1 - D_{t-1|r_{t-1}>0}) OMAA_{t-1} \end{aligned} \quad (23)$$

Model 2.7:

$$\log(h_t) = \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 \log(IV_{t-1}) + \beta_4 \log(|VRP_{t-1}|) \quad (24)$$

Model 2.8:

$$\begin{aligned} \log(h_t) = & \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 \log(IV_{t-1}) + \beta_4 \log(|VRP_{t-1}|) \\ & + \beta_5 D_{t-1|r_{t-1}>0} OMAA_{t-1} \end{aligned} \quad (25)$$

Model 2.9:

$$\begin{aligned} \log(h_t) = & \beta_0 + \beta_1 |\varepsilon_{t-1}/\sqrt{h_{t-1}}| + \beta_2 \log(h_{t-1}) + \beta_3 \log(IV_{t-1}) + \beta_4 \log(|VRP_{t-1}|) \\ & + \beta_5 (1 - D_{t-1|r_{t-1}>0}) OMAA_{t-1} \end{aligned} \quad (26)$$

Models 2.1 to 2.9 are the exponential versions of Models 1.1 to 1.9. Model 2.1 is the vanilla EGARCH without ambiguity attitudes, and acts as a benchmark for Models 2.2 and 2.3 which consider OMAA under gains and losses respectively. In Models 2.4 to 2.6, we also take into account the impact of implied variance on subsequent conditional variance and investigate if the information from OMAA innovations is still

significant (Models 2.5 and 2.6). In Models 2.7 to 2.9, we further examine the information content of OMAA when both implied variance and variance risk premium are controlled for.

#### **2.4 *Estimation, Inference and Diagnostic Analysis***

We estimate the models described in the previous section by maximizing their log-likelihood functions. Inference of variables is based on robust t-statistics as described in Bollerslev and Woodridge (1992). In addition to the estimated coefficients and robust t-statistics, the likelihood ratio and its chi-squared test statistics are also reported to judge the significance of the added OMAA parameter. In-sample error statistics based on four loss functions (described in the next section) are also reported to evaluate the in-sample forecasting performance of each model. For diagnostic analysis, we examine whether the inclusion of option market ambiguity attitudes reduces the skewness and excess kurtosis of standardized residuals from the mean equations. Jarque-Bera test statistics are also reported to compare the normality of standardized residuals from each model.

#### **2.5 *Out-of-sample Forecasting***

We further test the consistency of the relationship between option market ambiguity attitude and conditional variance, and the ability of improving variance forecasts by assessing the out-of-sample forecasting accuracy of the models. To ensure the positivity of conditional variance forecasts out-of-sample, we carry out out-of-sample analyses based on Models 2.1 to 2.9. To compare the variance forecasts across models, an estimated variance benchmark is needed. This benchmark relates to a one-step ahead realized volatility computed using intra-day returns. Due to the high degree of noise in using daily squared returns as the variance benchmark (Andersen et al. 2001; Andersen et al. 2003), we employ a model-free realized variance computed from intraday 1-minute return data. As evidence suggests that the realized variance computed from 1-minute return data can be noisy due to intraday market microstructure issues, we also compute realized variance with rolling 5-minute and 10-minute squared return on 1-minute grid (e.g., Stroud and Johannes 2014). This approach follows Christoffersen et al. (2014) by minimizing the noise while preserving all the information subsumed in 1-minute returns.

Following Chou (2005), Wei et al. (2010), and Hou and Suardi (2011), we consider the following four loss functions (LF) to evaluate the performance of each of the variance models.

*LF 1: Root mean squared error (RMSE)*

$$RMSE_M = \sqrt{\frac{1}{T} \sum_{t=1}^T (EV_{t+1} - FV_{M,t+1})^2} \quad (27)$$

*LF 2: Mean absolute error (MAE)*

$$MAE_M = \frac{1}{T} \sum_{t=1}^T |EV_{t+1} - FV_{M,t+1}| \quad (28)$$

*LF 3: Logarithmic loss function (LL)*

$$LL_M = \frac{1}{T} \sum_{t=1}^T \left[ \ln \left( \frac{EV_{t+1}}{FV_{M,t+1}} \right) \right]^2 \quad (29)$$

*LF 4: Loss implied by Gaussian likelihood (QLIKE)*

$$QLIKE_M = \frac{1}{T} \sum_{t=1}^T \left[ \ln(FV_{M,t+1}) + \frac{EV_{t+1}}{FV_{M,t+1}} \right] \quad (30)$$

where  $EV_{t+1}$  represents the estimated variance benchmark as measured by our intraday variance proxies at time  $t+1$ ,  $FV_{M,t+1}$  represents the forecasted variance from model  $M$  in which  $M$  stands for each of the Models 2.1 to 2.9. While RMSE is the most commonly used loss function in variance forecasting, MAE and QLIKE have been shown to be more robust (Fan et al. 2008; Wei et al. 2010; Hou and Suardi 2011). More importantly, QLIKE measures relative forecasting error and is known to penalize more heavily forecasts that underestimate the benchmark estimated variance. In this paper, we mainly rely on QLIKE in drawing our conclusions regarding the accuracy of our ambiguity-based GARCH volatility forecasts. QLIKE is indeed preferable for risk management and investment purposes. From a risk management standpoint, underestimating volatility may cost more than overestimating it by the same amount. From an investment point of view, relative forecasting errors are more relevant than absolute forecasting errors as returns are computed relative to an investment cost. We also implement the test for superior predictive

ability (SPA)<sup>7</sup> using the bootstrapping method proposed by Hansen and Lunde (2005) as a reality check for data snooping issues.

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<sup>7</sup> A similar diagnostic technique is the reality check (RC) for data snooping of White (2000). Hansen and Lunde (2005) compared 330 GARCH-based models and found RC to be less powerful and fails to detect inferior models. Herein, we rely on the SPA test for out-sample model evaluation.



### 3. DATA AND VARIABLES

#### 3.1 *Option Data*

We employ a dataset of European S&P 500 index options for the period of 2 Jan 1990 to 31 Dec 2014. To ensure the liquidity of option contracts included, we follow the option contract selection method of the Chicago Board Options Exchange (CBOE) in computing the VIX (Chicago Board Options Exchange 2014). To confirm that the information extracted from traded option prices is not biased towards certain option specification, we include both call and put options<sup>8</sup> with different moneyness including out-of-the-money (OTM), at-the-money (ATM), and in-the-money (ITM). Similar to the CBOE, we include both near-term options and next-term options in our sample. The average day-to-maturity (DTM) of our near-term and next-term option samples are 19.13 days and 49.16 days respectively. Our sample contains 1,116,326 contract-days with an average moneyness (S/K) of 1.06, and an average DTM of 33.99 days. Given the large scale of the dataset, we report basic summary statistics with option data sorted by DTM, VIX level, and the degree of OMAA. This helps us understand relationships among implied volatility, ambiguity attitudes, and option contract specification.

Table 1 summarizes the key characteristics of the option data used, including the number of option prices / contract-days, average price, average Black Scholes implied volatility, average bid-ask spread, and average bid-ask spread to average price ratio. From Panel C of Table 1, we can observe that when investors are extremely ambiguity seeking ( $0.75 \leq \text{OMAA}$ ) or very ambiguity averse ( $\text{OMAA} < 0.25$ ), the average bid-ask spread to price ratio tends to be higher. Taking the percentage bid-ask spread as a proxy for illiquidity, our data confirms a positive association between market illiquidity and ambiguity in line with extant research on market microstructure (Routledge and Zin 2009) and linkages between ambiguity and liquidity (Jiang et al. 2014).

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<sup>8</sup> According to the CBOE selection methodology, call and put options have matched strikes meaning there are equal numbers of call and put option contracts selected at any given moment. As a robustness check, we also extracted OMAA from only calls and then only puts. Our results are robust regardless of using OMAA from call options, put options prices or an average of both.

<Table 1>

### 3.2 *Stock Market Data*

Daily closing levels of the S&P 500 index, daily 1-month USD LIBOR, and the daily trailing twelve month dividend yield of the S&P 500 index were obtained from Thomson Datastream. Data of U.S. 3-month Treasury bill rate was obtained from the Federal Reserve Bank of St. Louis.

To assess the out-of-sample forecasting performance of our models, intraday high frequency return data was used to compute the realized variance based on different sampling grids. This intraday dataset consists more than 2.2 million data points from 2 January 1990 to 31 Dec 2014. We construct three realized variance measures as the estimated variance used in our out-of-sample performance evaluations. Realized variance based on 1-minute returns is simply the sum of squared 1-minute returns during the day. As noted in the literature (Martens and van Dijk 2007; Bandi et al. 2008; Christoffersen et al. 2014), the 1-minute realized variance can be noisy due to market microstructure effects, we also compute 5-minute and 10-minute realized variance with rolling grids as a result. For realized variance based on 5-minute returns, we start computing realized variance as the sum of squared 5-minute returns from the first price on 1-minute grid. Once finished with the rolling approach from the first minute of the day, we compute the realized variance starting from the second minute price of the day. We repeat these steps until we have 5 realized variance estimates in a day and take the sample average to obtain the 5-minute realized variance.

<Figure 1>, <Table 2>

The daily returns on the S&P 500 index, realized variance (measured as squared realized volatility) from the average RV estimator based on 5-minute intraday returns, daily levels of VIX, variance risk premium and option market ambiguity attitudes are plotted in Figure 1. The Great Financial Crisis of 2007-2008 dominates the picture in Graph A. The realized variance plot in Graph B is characterized by two episodes

of high volatility: a mini-crash<sup>9</sup> in Oct 1997 and the crisis of 2007-2008. A prolonged low volatility era from 2003 to 2006 is also evident in Graph B. From Graph D, we can see that 2007-2008 is dominated by ambiguity seeking behavior (where  $OMAA > 0.5$ ). OMAA recorded its maximum value of 0.8191 on 21 Oct 2008 and minimum value of 0.2196 on 28 Oct 1997, the day after the mini-crash. While OMAA has a daily average of 0.5313 over the entire 25-year sample period, ambiguity seeking behavior is most prominent in the crisis year of 2008. Daily average OMAA in 2008 is 0.5870, the highest average level in the 25 years covered.

Table 2 reports the descriptive statistics and correlation matrix for excess returns, implied variance, variance risk premium, option market ambiguity attitudes, and the three realized variances specified above. In Panel A of Table 2, we observe that the long-run mean excess return of the market is close to zero. Excess returns are negatively skewed and have excess kurtosis. On the other hand, average OMAA is at 0.5313, just above the ambiguity neutral threshold (0.5). This implies investors in the S&P 500 index option market are on average moderately ambiguity seeking<sup>10</sup>. Panel B of Table 2 shows that the correlations of OMAA with IV and VRP are very low at 0.08 and 0.039 respectively. This provides preliminary evidence that the information content of OMAA is distinct from that of IV and VRP.

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<sup>9</sup> The Mini-crash refers to the stock market crash of October 27, 1997 that is believed to have been caused by the economic crisis in Asia.

<sup>10</sup> OMAA is moderately correlated (correlation = 0.26) with the United States Valuation Index developed by the Yale School of Management, indicating investors' ambiguity seeking is positively related to their confidence / aggressiveness in valuing stocks.

## 4. GARCH VOLATILITY FORECASTING AND THE ROLE OF AMBIGUITY ATTITUDES

This section presents our empirical results examining the relationship between ambiguity attitudes and ex post conditional variance in the gain/loss domains. As noted, we assess the informational efficiency of OMAA by comparing our ambiguity-based models to benchmark models (without ambiguity) under GARCH-in-mean and also via the exponential GARCH-in-mean to ensure positivity of the conditional variance.

<Table 3>

### 4.1 *GARCH-in-mean Estimation and In-sample Forecasting*

Table 3 presents our estimates for Models 1.1 to 1.9. Model 1.1 is the vanilla GARCH model (without ambiguity) used as a benchmark for comparison with Models 1.2 and 1.3 (with ambiguity), while Model 1.4 serves as the benchmark model involving IV as an exogenous variable for comparison with Models 1.5 and 1.6 containing OMAA innovations. From Models 1.1 to 1.3, it is clear that ambiguity innovations provide additional prediction information and have different effects on conditional variance for gains and losses. In Model 1.2, innovations from ambiguity in the gain domain result in downward revisions of the conditional variance. Turning to the loss specification (Model 1.3), innovations from ambiguity generate positive revisions to the conditional variance. In both cases, judging by the reported likelihood ratios, ambiguity attitudes information is significant in explaining ex post variations in conditional variance. Without controlling for the effect of implied variance in the model, ambiguity attitudes matter more to the revisions of conditional variance in the domain of losses. In general, from Models 1.2 and 1.3, we learn that increased ambiguity seeking is associated with upward revisions to the conditional variance in the loss domain while increased ambiguity aversion is associated with upward revisions of the conditional variance in the gain domain. This is directly in line with the qualitative prescriptions of the hope and fear effects.

In Models 1.4 to 1.6, we consider implied variance as one of the explanatory variables in the variance equations. When compared to Models 1.1-1.3, Models 1.4 to 1.6 show improved fit, as measured by the log

likelihood, when implied variance is included. In Model 1.5, OMAA is significant and once again negatively related to conditional variance in the gain domain. The LR statistic confirms that the inclusion of OMAA is meaningful and significant. The negative relationship between OMAA and conditional variance in the gain domain is improved by the inclusion of implied variance. This confirms the result obtained in Model 1.2. Option market ambiguity attitudes are robustly correlated with ex post conditional variance in the gain domain. Turning to Model 1.6, the OMAA coefficient in the loss domain is positive and significant with an improved robust t-statistic of 3.191. The log-likelihood of Model 1.6 reveals a better explanatory power for OMAA in the loss domain (than under gains) when implied variance is controlled for. Further considering the effect of OMAA on the variance process in the presence of implied variance (i.e., risk-neutral expectations) and the variance risk premium, a proxy for risk aversion, we turn to Models 1.7 to 1.9. Our results remain consistent when the variance risk premium is controlled for. As in Models 1.1 to 1.6, relationships between OMAA and future variance remain positive (negative) in the loss (gain) domain.

Considering all models in Table 3 collectively, our study unveils the important role of ambiguity attitudes in the conditional variance process. Judging by the log likelihood output, Model 1.9 is the most successful in terms of in-sample forecasting accuracy and efficiency<sup>11</sup>. Our results overall suggest that option market ambiguity attitudes capture important information regarding future evolutions of conditional volatility and are efficient in improving the accuracy of GARCH volatility forecasts in practice.

Turning to the in-sample error statistics in Panel B of Table 3, for models without implied variance as an exogenous variable (Models 1.1 to 1.3), all error statistics suggest that the model based on losses (Model 1.3) produces the best volatility forecasts. Considering all error statistics at the same time suggests that accounting for OMAA improves the accuracy of the in-sample volatility forecasts. For Models 1.4 to 1.6, with implied variance as an exogenous variable, the error statistics of Models 1.5 and 1.6 beat those of

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<sup>11</sup> For robustness, we also included a GJR term for asymmetric shocks and found OMAA remains significant in the variance equations. With the inclusion of asymmetric shocks in Models 1.5(1.6), OMAA remains significant and negatively (positively) associated with ex post variance. The coefficients of OMAA are generally more significant than those of GJR. The GJR term is insignificant under EGARCH, and thus not suitable for out-of-sample comparison.

Model 1.4. Regarding Models 1.7-1.9, error statistics of Models 1.8 and 1.9 beat those of Model 1.7. The in-sample error statistics support the model estimation results and suggest that option market ambiguity attitude is quantitatively important in determining ex post conditional volatility. This sheds light on how to improve the accuracy of GARCH volatility forecasts by considering investors' attitudes to uncertainty in prediction exercises.

<Table 4>

Given the non-normality property of US stocks return data, a good variance model should also reduce, or ideally remove, the negative skewness and excess kurtosis in the residuals (Campbell and Hentschel 1992). To further understand the importance of ambiguity aversion in variance modeling, we carry out additional diagnostic tests to check the impact of ambiguity attitudes on conditional variance forecasting while considering skewness and kurtosis dynamics. Table 4 reports diagnostic tests on the standardized residuals from the mean equations of Models 1.1 to 1.9. Table 4 shows that skewness and excess kurtosis levels are reduced in all models containing the OMAA measure when compared to their corresponding benchmark counterparts. Although the t-statistics still suggest that skewness and excess kurtosis cannot be ruled out, reductions in skewness and excess kurtosis confirm the findings in Table 3 that with the inclusion of ambiguity attitudes in the variance processes, better model fits can be obtained. Jarque-Bera statistics that summarize information from skewness and excess kurtosis dynamics further confirm that the residuals from the models with ambiguity are much closer to normality.

The success of models with OMAA innovations renews our understanding of the economic value of accounting for model uncertainty and investors' ambiguity attitudes in conditional variance estimations. Results from this section lead to the following conclusions on the impact of OMAA on conditional variance:

1. The relationship between conditional variance and option market ambiguity attitude is negative and statistically significant in the gain domain, meaning increases in ambiguity aversion is related to increases in conditional variance.

2. The relationship between conditional variance and option market ambiguity attitude is positive and statistically significant in the loss domain, meaning increases in ambiguity seeking is related to increases in conditional variance.
3. In general the inclusion of OMAA allows better fits of empirical data and that conclusions 1-2 hold after controlling for option implied variance (risk-based option information) and the variance risk premium (risk aversion information).

The next section turns to the exponential versions of Models 1.1 to 1.9 for robustness and also as a prelude to the out-of-sample forecasting analysis (covered in Section 4.3).

<Table 5>

#### **4.2 Exponential GARCH-in-mean Estimation and In-sample Forecasting**

Table 5 reports the estimation results for Models 2.1 to 2.9. Model 2.1 is the benchmark model without ambiguity, implied variance, or variance risk premium. Comparing Models 2.1 and 2.2, OMAA towards gains exhibits the same negative relationship with conditional variance as in the previous section. The robust t-statistic of OMAA in the gain domain is -4.887, which suggests that OMAA is more significant in Model 2.2 than in Model 1.2. Model 2.3 shows a positive and statistically significant relationship between OMAA and conditional variance in the loss domain, confirming our findings from the previous section. In general, the OMAA measure is statistically significant for both gains and losses, with the likelihood ratio being significant at the 99.99% confidence level.

Turning to the specifications with controls for implied variance, Models 2.4 to 2.6 show similar conclusions as those in Models 1.4 to 1.6. From Model 2.5, OMAA towards gains exhibits a negative relationship with conditional variance but with a slightly weaker robust t-statistics when compared to Model 2.2. On the other hand, in Model 2.6, OMAA towards losses shows a strong positive relationship with ex post conditional variance. OMAA is significant under gains and losses. In general, the significance of OMAA is improved when the models are specified in exponential form (when compared to robust t-statistics in Table 3). With risk aversion (i.e., VRP) being controlled for, Models 2.7 to 2.9 show consistent

results for both gains and losses. Judging from the log likelihood of 20,773, Model 2.9 produces the best data fit across all specifications considered in this subsection.

Turning to the in-sample error statistics for Models 2.1 to 2.9 in Panel B of Table 5, similar conclusions can be drawn: the inclusion of option market ambiguity attitudes does improve the accuracy of GARCH volatility forecasts. When we consider the in-sample error statistics of Models 2.1 to 2.9, RMSE and MAE tend to give mixed suggestions as to which model gives the most accurate forecasts. In light of that, and as noted earlier, we rely on QLIKE, which has proved to be more reliable in assessing the quality of forecasts (e.g., Kumar 2015), for model comparison. The in-sample QLIKE for Model 2.9 gives  $0.4985 \times 10^3$ , the smallest error in all nine specifications (Table 5). The results from Table 5 confirm our findings and highlight the suitability of using exponential models in forecasting exercises.

<Table 6>

In line with Table 4, we present the diagnostic tests results for Models 2.1 to 2.6 in Table 6. Skewness and excess kurtosis of the standardized residuals from the mean equations are in general reduced in all models involving OMAA except for skewness in Model 2.2. This suggests that models with OMAA tend to better account for the non-normality of S&P 500 returns. Standardized residuals from Model 2.9 have the least skewness and least excess kurtosis out of the nine models. The magnitude of skewness is reduced from -0.413 in benchmark Model 2.1 to -0.340 in Model 2.9. Excess kurtosis decreases from 1.728 in Model 2.1 to 1.216 in Model 2.9. Jarque-Bera statistics in Table 6 also suggest that the standardized residuals from the various models with OMAA innovations are much closer to normality.

The in-sample forecasting success of option market ambiguity helps us understand the asymmetric property of OMAA in affecting revisions in conditional variance. Our analysis is the first empirical study to reveal this special relationship in financial markets data in the context of the GARCH volatility framework. The approach of asymmetric modeling of conditional variance with ambiguity information provides insightful and statistically significant explanatory power beyond what traditional variance models offer. Despite that, the economic value of including OMAA in variance forecasting might also depend on



the model and parameter stability. To remedy this, the next section examines the out-of-sample forecasting ability of our ambiguity-based models to further corroborate our conclusions.

### 4.3 *Out-of-sample Volatility Forecasting*

While the in-the-sample analysis in the previous sections allowed us to understand the empirical implications of option market ambiguity attitudes for GARCH volatility modeling, the validity of the improved variance models as an economically valuable tool for market statisticians also depends on the out-of-sample forecasting performance. Herein, we examine the out-of-sample forecasting performance of Models 2.1 to 2.9 over a period of 20 years from 1995 to 2014. To ensure the stability of model parameters, we employ at least 5 years of data in the estimation. Conditional variance forecasts are then compared to realized variance with different intra-day estimation grids (1, 5 and 10 min grids as mentioned in Section 3). We employ four loss functions / error statistics with respect to the measured variance for the evaluation of the forecasting ability of each model. In addition to RMSE and MAE, the most traditional error statistics used in out-of-sample forecasting evaluation, we once again consider LL, which measures the logarithmic forecasting accuracy, and QLIKE which measures the relative forecasting accuracy and penalizes underestimated variance forecasts. As noted, due to the important properties of QLIKE in risk management and investment, our conclusions mainly rely on the QLIKE statistics. As discussed, we also compare the various models using Hansen (2005)'s superior predictive ability (SPA) test.

<Table 7>

Table 7 reports the error statistics from the out-of-sample forecasting analysis in Models 2.1 to 2.9 based on a 5-year (1250 trading days) rolling estimation window. Hansen's SPA test bootstrapped p-values are reported in parentheses. Since we have 6296 daily observations from Jan 1990 to Dec 2014, error statistics in Table 7 are based on a 5046 days out-of-sample forecasting window.

The first three columns of Table 7 report out-of-sample forecasting errors for models (M2.1 to 2.3) that do not involve implied variance as an exogenous variable, the middle three columns report the errors based on models (M2.4 to 2.6) with implied variance in the variance process, and the last three columns report

the errors based on models (M2.7 to 2.9) with both implied variance and variance risk premium included in the variance process. Forecasting errors that beat the benchmarks are reported in bold for easier comparison. From Table 7, comparing loss functions in Models 2.1-2.3 points to Model 2.2 as the best out-of-sample forecasting model with the lowest errors in a majority of loss functions and RV computation grid. Model 2.3 dominates QLIKE statistics and appears to be the best for risk management purposes. According to the LL and QLIKE criteria, models with option market ambiguity attitudes consistently have smaller out-of-sample forecasting errors when compared to the benchmark Model 2.1. SPA test p-values suggest that Model 2.2 is the best model when data snooping bias is accounted for. Turning to Models 2.4 to 2.6, which take into account the impact of ex ante implied variance, models (M2.5 and 2.6) containing OMAA innovations produce better variance forecasts when compared to the benchmark Model 2.4 without ambiguity. LL and QLIKE point to Model 2.5 as the best model to forecast conditional volatility. SPA test p-values also clearly point to Model 2.5 as the best forecasting model overall. Interestingly but unsurprisingly when we compare out-of-sample forecasting errors under RMSE and MAE, the inclusion of implied variance actually increases the absolute value of forecasting errors. Alternatively when we assess models using LL and QLIKE statistics, the inclusion of implied variance does help avoid the under-estimation of conditional variance problem. Turning to Models 2.7-2.9, the inclusion of OMAA reduces the out-of-sample forecast errors in all cases. SPA tests also lead to similar conclusions with a majority of forecasts displaying statistical significance. From Table 7, it is clear that the inclusion of OMAA in the variance models significantly improves the accuracy of forecasts for different realized variance specifications and considering various loss functions.

The significant out-of-sample results highlight the efficiency of option market information and ambiguity attitudes in forecasting volatility. To further appreciate the economic significance of option market ambiguity attitudes in financial markets and volatility modeling, the next section studies the potential economic gains made from trading strategies based on our out-of-sample GARCH volatility forecasts from Models 2.1 to 2.9.

#### 4.4 *Economic Significance Analysis*

We extend the analysis to verify if investors can benefit from more accurate variance forecasts and generate excess profits based on the proposed forecasting models (Fleming et al. 2003; Marquering and Verbeek 2004; Boguth et al. 2011). Similar to Han (2006) and Driesprong et al. (2008), we implement a daily-rebalancing volatility timing strategy based on the level of variance forecasts relative to the historical average realized variance. We consider a risk-seeking investor<sup>12</sup> who invests in the S&P 500 portfolio whenever the next day variance forecast is exceeding the two-week average of realized variance. At the end of each trading day, (s)he will decide whether to buy (or sell if the position was already established) the portfolio. With this strategy simply consisting of “buy” and “sell” timing without accounting for portfolio reallocation, we assess the economic significance of OMAA variance models in market timing. To allow for a more flexible trading strategy, we also consider an alternative situation in which the investor can go short when the variance forecast is lower than the historical average. In the economic significance analysis, we use out-of-sample forecasts from each model as the basis of the trading signal. The trading window in which the strategies are implemented therefore covers the 1995-2014 period. Following Driesprong et al. (2008), we assume a conservative transaction cost<sup>13</sup> of 0.1% each way for “buy” and “sell”.

<Table 8>

Table 8 reports the portfolio performance of the volatility timing strategy under each forecasting model. The benchmark for Models 2.2 and 2.3 is Model 2.1, for Models 2.5 and 2.6 is Model 2.4, and for Models 2.8 and 2.9 is Model 2.7. From Panel A, under the long-only strategy over the full period, ambiguity-based volatility Models 2.2 and 2.3 generate 33.43 and 32.63 percentage points return in excess of the benchmark

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<sup>12</sup> According to Merton (1973), risk is positively related to expected returns. Since we consider the case of one-asset (S&P 500 index) volatility timing strategy, it is logical to assume investors who would follow this strategy to be risk-seeking.

<sup>13</sup> Research on the economic significance of volatility timing trading strategies adopts various transaction costs assumptions. Bhardwaj and Brooks (1992), and Balduzzi and Lynch (1999) suggest 0.5% for individual equity trading; Driesprong et al. (2008) suggest that 0.1% is more reasonable for futures trading on commodities; and Fleming et al. (2003) suggest 0.01% for futures trading on equities. Investors can easily invest in index tracking ETFs, such as SPDR SPY which has a correlation of 99% with S&P500 index, and incur very low transaction costs. For example the largest electronic brokerage house in the U.S. charges only \$0.005 per share for SPY transactions, which amounts to a transaction cost of 0.002% (as of Sept 2016). Although investors can trade the index cheaply by various ETF and index futures, we employ a more conservative rate of 0.1% each way as suggested by Driesprong et al. (2008).

(i.e., Model 2.1 without ambiguity innovations). Turning to Model 2.4 and its variants, which incorporate information from implied variance, ambiguity-based volatility Models 2.5 and 2.6 outperform the benchmark (Model 2.4 without ambiguity) by 58.77 and 29.34 percentage points over the whole period. For models that incorporate information from implied variance, variance risk premium and OMAA, Models 2.8 and 2.9 deliver some impressive excess returns of 128.46 and 83.17 percentage points relative to their benchmark, respectively. All strategies based on models with ambiguity innovations generate higher returns than the benchmark models. This suggests that, when used as part of a volatility timing strategy, the inclusion of investors' ambiguity attitudes in volatility forecasting exercises can help yield superior returns. Turning to the sub-period results in Panel A, the superior returns generated by the volatility-timing strategies involving OMAA generally hold during (and after) the financial crisis of 2008 and the collapse of Lehman Brothers. While these strategies tend to produce higher returns than the benchmark in 1995-2004, their returns are even more impressive in the 2005-2014 follow-up period. Model 2.8, which considers option market ambiguity attitudes in the gain domain together with implied variance and risk aversion, produces a striking return of 95.46% during 2005-2014. This is equivalent to 42.55 percentage points in excess of the benchmark Model 2.7.

To further appreciate the importance of option market ambiguity attitudes in volatility-timing investment strategies, we consider a more flexible case which allows the investors to go short when the forecasted volatility is below the historical average level. Since the long-short strategies ensure the investor to have a continuous exposure to the S&P 500 index (either short or long at a given time), volatilities of the portfolio values are identical to those of the S&P 500 index<sup>14</sup>. In Panel B of Table 8, we can observe that the long-short strategy relying on GARCH volatility forecasts from our ambiguity Model 2.8 generates an astonishing 610.31% return over our entire sample period of 1995-2014 compared to only 296.45% for the benchmark (Model 2.7). Considering the full period performance of the various strategies in Panel B,

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<sup>14</sup> Slight discrepancies in portfolio volatilities as shown in Panel B are due to the transaction costs. In the case of a perfect market without transaction costs, the portfolio volatility based on long-short strategy in all models will be identical.

ambiguity-based GARCH volatility models produce superior returns than the benchmarks. In 1995-2014, all ambiguity-based strategies with the inclusion of implied variance and risk aversion generated between 52.44 and 99.68 percentage points in excess of the corresponding benchmark or unambiguous models. Similar to the long-only strategy, our models in general perform equally well (and outperform benchmarks) during more uncertain periods.

The superior performance of our ambiguity-based strategies highlights the economic value of option market ambiguity attitudes in GARCH volatility forecasting, risk management, and market timing. The economic significance analysis confirms that the proposed volatility forecasting models not only provide more accurate volatility forecasts both in-sample and out-of-sample, but are also important in market-timing especially during periods of economic uncertainty.

## 5. CONCLUSION

We propose an ambiguity-based GARCH-in-mean volatility forecasting framework which capitalizes on the asymmetric and well documented role of market investors' ambiguity attitudes in the gain and loss domains. Our empirical results, based on a comprehensive dataset that spans 25 years of option pricing data and containing 6296 daily observations, confirm a significant relationship between ambiguity attitudes and ex post conditional volatility, and suggest that ambiguity and model uncertainty, as inferred from the option market, are quantitatively important in determining ex post conditional variance and in improving forecasts.

We evaluate the practical relevance of our ambiguity-based volatility forecasting models by examining their out-of-sample forecasting accuracy. Out-of-sample forecasting analysis using the SPA test, different estimation windows, comparison windows, estimated volatility benchmarks, and an extensive range of loss functions documents robust and consistent improvements in the accuracy of GARCH volatility forecasts when investors' ambiguity attitudes, implied variance, and investors' risk aversion are controlled for. In addition to the out-of-sample analysis, we also examine the economic value of including option market ambiguity attitudes in volatility forecasting practice by comparing portfolio returns based on two simple volatility-timing trading strategies with transaction costs. Economic significance results confirm the findings of the in-sample and out-of-sample estimations, and show that an annualized return of 3.2% in excess of the benchmark portfolio return can be earned using the volatility forecasts from our proposed models.

Given the consistent findings from the in-sample estimations, out-of-sample forecasting, and economic significance analyses, option market ambiguity attitudes can be considered a statistically significant and quantitatively important factor in GARCH volatility forecasting. This paper validates the hypothesis of efficiency of options markets in informing future volatility fluctuations and serves as a foundation for research on the role of ambiguity aversion and model uncertainty in other risk management areas such as value at risk analysis and stress testing. We leave this for future research.

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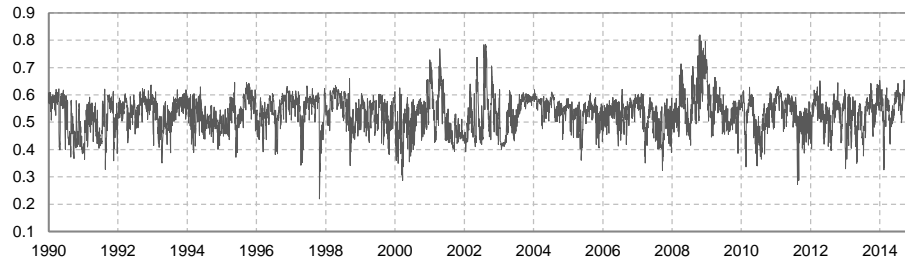
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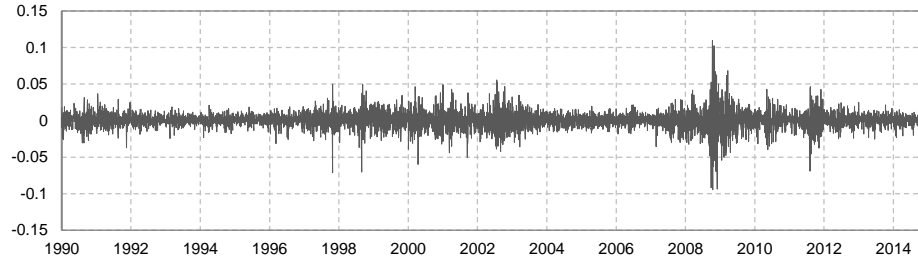
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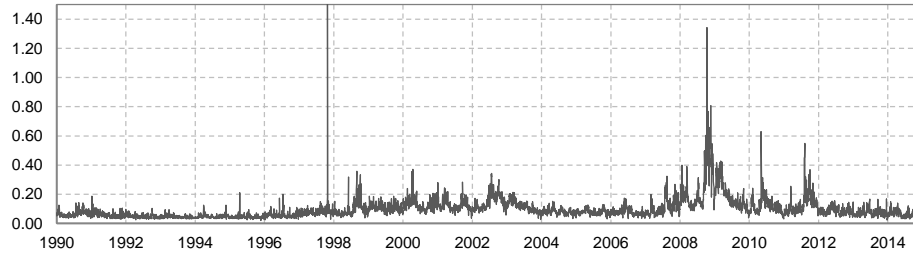
*Graph A. Daily Ambiguity Attitudes from S&P 500 Options (OMAA)*



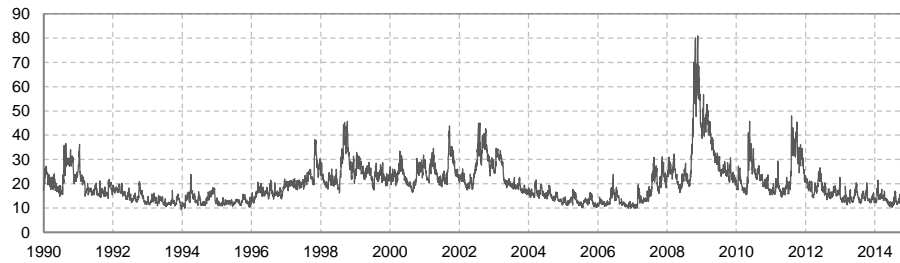
*Graph B. Daily S&P 500 Returns*



*Graph C. Daily Realized Volatility from Average RV Estimator (5-min)*



*Graph D. Daily Level of VIX*



*Graph E. Daily level of Variance Risk Premium (VRP)*

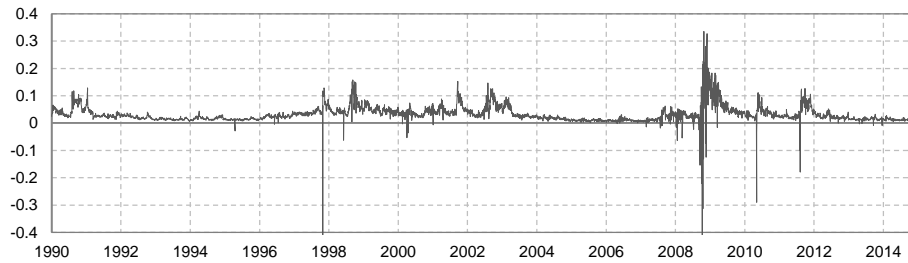


Figure 1. Option Market Ambiguity Attitude, S&P 500 Daily Returns, Realized Volatility, VIX, and variance risk premium. Graph A plots the daily ambiguity attitudes from S&P 500 options. Graph B plots the daily S&P 500 returns. Graph C plots the daily realized volatility from an average RV estimator with rolling 5-minute squared return on 1-minute grid. Realized volatility in Graph C is shown in annualized standard deviation terms. Graph D plots the daily VIX index. Graph E plots the daily variance risk premium computed as the difference between implied variance and realized variance. All plots span 2 Jan 1990 to 31 Dec 2014.

Table 1. Summary Statistics for Option Data

<i>Panel A. By Day-to-maturity (DTM)</i>						
	<i>DTM &lt; 14</i>	<i>14 ≤ DTM &lt; 30</i>	<i>30 ≤ DTM &lt; 45</i>	<i>45 ≤ DTM &lt; 60</i>	<i>60 ≤ DTM</i>	<i>All</i>
Number of Contract-days	182,924	342,276	272,966	182,418	135,742	1,116,326
Average Price	46.625	62.982	69.481	68.549	73.703	64.104
Average Implied Volatility	0.296	0.261	0.251	0.236	0.189	0.252
Average Bid-ask Spread	1.806	1.869	1.916	1.814	2.079	1.887
Average Bid-ask Spread / Price	3.87%	2.97%	2.76%	2.65%	2.82%	2.94%
<i>Panel B. By level of VIX</i>						
	<i>VIX &lt; 15</i>	<i>15 ≤ VIX &lt; 20</i>	<i>20 ≤ VIX &lt; 25</i>	<i>25 ≤ VIX &lt; 30</i>	<i>30 ≤ VIX</i>	<i>All</i>
Number of Contract-days	313,424	314,978	238,112	114,218	135,594	1,116,326
Average Price	53.679	59.613	68.203	73.241	83.740	64.104
Average Implied Volatility	0.149	0.215	0.273	0.322	0.475	0.252
Average Bid-ask Spread	1.608	1.813	1.909	1.980	2.583	1.887
Average Bid-ask Spread / Price	3.00%	3.04%	2.80%	2.70%	3.08%	2.94%
<i>Panel C. By level of OMAA</i>						
	<i>OMAA &lt; 0.25</i>	<i>0.25 ≤ OMAA &lt; 0.5</i>	<i>0.5 ≤ OMAA &lt; 0.75</i>	<i>0.75 ≤ OMAA</i>	<i>All</i>	
Number of Contract-days	194	299,006	810,426	6,700	1,116,326	
Average Price	59.275	62.958	64.210	102.545	64.104	
Average Implied Volatility	0.309	0.263	0.245	0.551	0.252	
Average Bid-ask Spread	2.988	1.858	1.881	3.793	1.887	
Average Bid-ask Spread / Price	5.04%	2.95%	2.93%	3.70%	2.94%	

Table 1 reports the summary statistics of option data used. Panel A, B and C report the basic descriptive statistics sorted by day-to-maturity (DTM), level of VIX, and level of OMAA respectively. The data period covers 2 Jan 1990 to 31 Dec 2014.

Table 2. Descriptive Statistics and Correlation Matrix

<i>Panel A. Descriptive Statistics</i>							
	<i>ER</i>	<i>IV</i>	<i>OMAA</i>	<i>RV<sub>1min</sub></i>	<i>RV<sub>5min</sub></i>	<i>RV<sub>10min</sub></i>	<i>VRP</i>
<i>Mean</i>	0.0001	0.0002	0.5313	0.0001	0.0001	0.0001	0.0001
<i>Median</i>	0.0004	0.0001	0.5400	0.0000	0.0000	0.0000	0.0001
<i>SD</i>	0.0114	0.0002	0.0644	0.0002	0.0002	0.0002	0.0002
<i>Skewness</i>	-0.2338	4.8280	-0.1625	36.7055	37.9091	36.0445	-42.6582
<i>Kurtosis</i>	8.7475	36.5229	1.0297	1903.0536	2009.4099	1847.7728	2627.4863
<i>AR(1)</i>	-0.0562	0.9710	0.8268	0.3182	0.3042	0.3232	0.1725
<i>Panel B. Correlation Matrix</i>							
	<i>ER</i>	<i>IV</i>	<i>OMAA</i>	<i>RV<sub>1min</sub></i>	<i>RV<sub>5min</sub></i>	<i>RV<sub>10min</sub></i>	<i>VRP</i>
<i>ER</i>	1.000						
<i>IV</i>	-0.133	1.000					
<i>OMAA</i>	0.002	0.080	1.000				
<i>RV<sub>1min</sub></i>	-0.018	0.511	0.030	1.000			
<i>RV<sub>5min</sub></i>	-0.016	0.502	0.028	0.999	1.000		
<i>RV<sub>10min</sub></i>	-0.019	0.516	0.032	0.999	1.000	1.000	
<i>VRP</i>	-0.098	0.312	0.039	-0.656	-0.665	-0.652	1.000

Table reports the descriptive statistics of key variables. Panel A reports the basic descriptive statistics for excess returns (ER), implied variance (IV), options market ambiguity attitudes (OMAA), realized variances (RV) using different estimation grids, and variance risk premium (VRP). Panel B reports the correlation matrix. The data period covers 2 Jan 1990 to 31 Dec 2014.

Table 3. GARCH-In-Mean Estimates of the Daily Ambiguity-Volatility Relation

<i>Panel A. Estimation Results</i>									
	<i>Model 1.1</i>	<i>Model 1.2</i>	<i>Model 1.3</i>	<i>Model 1.4</i>	<i>Model 1.5</i>	<i>Model 1.6</i>	<i>Model 1.7</i>	<i>Model 1.8</i>	<i>Model 1.9</i>
$\alpha_0$	1.84E-04 (1.233)	5.23E-05 (0.344)	1.00E-05 (0.066)	3.31E-05 (0.213)	2.89E-06 (0.019)	1.33E-05 (0.087)	5.22E-05 (0.377)	9.22E-05 (0.691)	1.02E-04 (0.762)
$\alpha_1$	3.211 ** (2.006)	3.854 ** (2.343)	3.933 ** (2.374)	0.874 (0.456)	1.266 (0.661)	1.187 (0.622)	0.481 (0.299)	0.242 (0.152)	0.175 (0.110)
$\beta_0$	1.17E-06 *** (4.322)	3.62E-06 *** (3.660)	-1.59E-06 ** (-2.517)	-1.19E-05 *** (-3.683)	-3.74E-07 (-0.094)	-1.22E-05 *** (-5.588)	-4.70E-06 * (-1.742)	3.56E-06 (1.111)	-8.30E-06 *** (-5.071)
$\beta_1$	0.080 *** (8.954)	0.079 *** (8.843)	0.073 *** (8.715)	-0.014 (-0.619)	0.004 (0.217)	0.005 (0.292)	-0.028 (-0.678)	-0.024 (-0.595)	-0.023 (-0.586)
$\beta_2$	0.911 *** (108.175)	0.907 *** (102.318)	0.914 *** (108.865)	0.011 (0.062)	0.318 *** (2.630)	0.340 *** (2.924)	0.073 (0.992)	0.270 *** (3.165)	0.276 *** (3.318)
$\beta_{OMAA, Gain}$	- (-2.621)	-7.46E-06 *** (-2.621)	-	- (-3.760)	-2.18E-05 *** (-3.760)	-	- (-4.296)	-2.18E-05 *** (-4.296)	-
$\beta_{OMAA, Loss}$	- (4.499)	- (4.499)	1.19E-05 *** (4.499)	- (4.074)	- (4.074)	2.34E-05 *** (4.074)	- (4.328)	- (4.328)	2.27E-05 *** (4.328)
$\beta_{IV}$	- (5.788)	- (5.788)	- (5.788)	0.704 *** (5.788)	0.464 *** (5.088)	0.448 *** (5.140)	1.290 *** (12.002)	1.027 *** (8.411)	1.012 *** (8.524)
$\beta_{VRP}$	- (-9.035)	- (-9.035)	- (-9.035)	- (-9.035)	- (-9.035)	- (-9.035)	-0.905 *** (-9.035)	-0.738 *** (-7.605)	-0.723 *** (-7.670)
Log Likelihood	20,537.97	20,550.50	<b>20,572.45</b>	20,668.86	20,686.14	<b>20,689.28</b>	20,771.07	20,791.78	<b>20,793.00</b>
LR	-	25.06	68.96	-	34.56	40.84	-	41.42	43.86
$P(LR) > \chi^2$	-	5.56E-07	1.00E-16	-	4.13E-09	1.65E-10	-	1.23E-10	3.53E-11
<i>Panel B. In-sample Error Statistics</i>									
<i>(x1000)</i>	<i>Model 1.1</i>	<i>Model 1.2</i>	<i>Model 1.3</i>	<i>Model 1.4</i>	<i>Model 1.5</i>	<i>Model 1.6</i>	<i>Model 1.7</i>	<i>Model 1.8</i>	<i>Model 1.9</i>
RMSE	0.0638	0.0601	<b>0.0587</b>	0.0606	<b>0.0545</b>	0.0551	0.0524	<b>0.0479</b>	0.0488
MAE	0.0825	0.0807	<b>0.0800</b>	0.0733	0.0726	<b>0.0725</b>	0.0761	0.0758	<b>0.0756</b>
LL	1.7581	1.7601	<b>1.7447</b>	1.7245	1.7156	<b>1.7148</b>	1.4831	<b>1.4716</b>	1.4728
QLIKE	0.5502	0.5497	<b>0.5465</b>	0.5427	0.5385	<b>0.5380</b>	0.4938	0.4904	<b>0.4904</b>

Table 3 reports the estimation results of the GARCH-M models. Panel A reports the estimated parameters of Models 1.1 to 1.9 based on maximum likelihood estimation. Robust t-statistics reported in parentheses are computed based on Bollerslev and Wooldridge (1992). Panel B reports the in-sample error statistics. Model 1.1 is the benchmark model for Models 1.2 and 1.3. Model 1.4 is the benchmark model for Models 1.5 and 1.6. Model 1.7 is the benchmark model for Models 1.8 and 1.9. We estimate the models using daily close-to-close returns of S&P 500 index, implied variance, variance risk premium, and option market ambiguity attitudes for the period 2 Jan 1990 to 31 Dec 2014. \*\*\*, \*\*, and \* indicate 99%, 95%, and 90% confidence levels respectively.



Table 4. GARCH-In-Mean Diagnostic Tests of the Daily Ambiguity-Volatility Relation

	<i>Model 1.1</i>	<i>Model 1.2</i>	<i>Model 1.3</i>	<i>Model 1.4</i>	<i>Model 1.5</i>	<i>Model 1.6</i>	<i>Model 1.7</i>	<i>Model 1.8</i>	<i>Model 1.9</i>
<b>Skewness</b>	-0.420 (-13.060)	-0.417 (-12.950)	-0.393 (-12.208)	-0.414 (-12.861)	-0.384 (-11.946)	-0.375 (-11.651)	-0.351 (-10.911)	-0.303 (-9.424)	-0.295 (-9.178)
<b>Excess Kurtosis</b>	1.703 (26.462)	1.586 (24.634)	1.469 (22.820)	1.576 (24.490)	1.295 (20.119)	1.245 (19.334)	1.327 (20.619)	1.101 (17.111)	1.072 (16.657)
<b>Jarque-Bera</b>	946.399	841.765	727.936	831.597	595.029	553.798	591.426	414.716	393.097

Table 4 reports the diagnostic tests of Models 1.1 to 1.9. Skewness and excess kurtosis are the estimated skewness and excess kurtosis of the standardized residuals from the mean equation. Model 1.1 is the benchmark model for Models 1.2 and 1.3. Model 1.4 is the benchmark model for Models 1.5 and 1.6. Model 1.7 is the benchmark model for Models 1.8 and 1.9. We estimate the models using daily close-to-close returns of S&P 500 index, implied variance, variance risk premium, and option market ambiguity attitudes over the period 2 Jan 1990 to 31 Dec 2014.

Table 5. Exponential GARCH-In-Mean Estimates of the Daily Ambiguity-Volatility Relation

<i>Panel A. Estimation Results</i>									
	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
$\alpha_0$	1.47E-04 (0.934)	-3.03E-05 (-0.185)	-4.25E-05 (-0.284)	2.99E-05 (0.208)	8.60E-06 (0.062)	1.32E-05 (0.096)	2.59E-05 (0.199)	2.34E-05 (0.186)	3.77E-05 (0.303)
$\alpha_1$	3.565 ** (1.993)	3.169 * (1.667)	2.129 (1.285)	0.748 (0.456)	1.098 (0.677)	1.054 (0.662)	0.543 (0.398)	0.905 (0.662)	0.806 (0.605)
$\beta_0$	-2.55E-01 *** (-8.203)	-2.30E-01 *** (-7.003)	-3.30E-01 *** (-10.256)	1.58E+00 *** (4.117)	1.13E+00 *** (3.640)	7.42E-01 *** (2.600)	5.60E-01 ** (2.196)	3.84E-01 ** (2.113)	1.02E-01 (0.562)
$\beta_1$	0.170 *** (9.902)	0.165 *** (9.794)	0.149 *** (9.647)	-0.083 * (-1.933)	-0.038 (-0.978)	-0.021 (-0.579)	-0.084 * (-1.822)	-0.073 * (-1.773)	-0.065 * (-1.653)
$\beta_2$	0.986 *** (354.096)	0.983 *** (314.021)	0.985 *** (412.748)	-0.050 (-0.317)	0.297 ** (2.340)	0.383 *** (3.510)	0.216 ** (2.171)	0.417 *** (5.445)	0.452 *** (6.408)
$\beta_{OMAA, Gain}$	-	-1.92E-01 *** (-4.887)	-	-	-4.26E-01 *** (-4.599)	-	-	-4.67E-01 *** (-5.629)	-
$\beta_{OMAA, Loss}$	-	-	3.06E-01 *** (8.188)	-	-	4.92E-01 *** (5.374)	-	-	5.05E-01 *** (6.077)
$\beta_{IV}$	-	-	-	1.283 *** (6.678)	0.857 *** (5.306)	0.749 *** (5.407)	1.467 *** (9.083)	1.139 *** (8.671)	1.073 *** (8.797)
$\beta_{VRP}$	-	-	-	-	-	-	-0.560 *** (-9.000)	-0.481 *** (-9.259)	-0.454 *** (-9.218)
Log Likelihood	20,530.43	20,557.53	<b>20,596.80</b>	20,689.80	20,702.36	<b>20,706.10</b>	20,745.27	20,769.15	<b>20,773.01</b>
LR	-	54.2	132.74	-	25.12	32.6	-	47.76	55.48
$P(LR) > \chi^2$	-	1.81E-13	1.03E-30	-	5.39E-07	1.13E-08	-	4.82E-12	9.44E-14
<i>Panel B. In-sample Error Statistics</i>									
<i>(x1000)</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
RMSE	0.0636	0.0575	<b>0.0560</b>	0.0564	<b>0.0497</b>	0.0502	0.0529	<b>0.0459</b>	0.0470
MAE	0.0794	<b>0.0766</b>	0.0812	<b>0.0794</b>	0.0796	0.0800	0.0864	<b>0.0854</b>	0.0870
LL	1.7706	1.7676	<b>1.7362</b>	1.7042	1.6941	<b>1.6899</b>	1.5557	1.5275	<b>1.5239</b>
QLIKE	0.5566	0.5534	<b>0.5458</b>	0.5372	0.5331	<b>0.5320</b>	0.5079	0.4995	<b>0.4985</b>

Table 5 reports the estimation results of the EGARCH-M models. Panel A reports the estimated parameters of Models 2.1 to 2.9 based on maximum likelihood estimation. Robust t-statistics reported in parentheses are computed based on Bollerslev and Wooldridge (1992). Panel B reports the in-sample error statistics. Model 2.1 is the benchmark model for Models 2.2 and 2.3. Model 2.4 is the benchmark model for Models 2.5 and 2.6. Model 2.7 is the benchmark model for Models 2.8 and 2.9. We estimate the models using daily close-to-close returns of S&P 500 index, implied variance, variance risk premium, and option market ambiguity attitudes over the period 2 Jan 1990 to 31 Dec 2014. \*\*\*, \*\*, and \* indicate 99%, 95%, and 90% confidence levels respectively.

Table 6. Exponential GARCH-In-Mean Diagnostic Tests of the Daily Ambiguity-Volatility Relation

	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
<b>Skewness</b>	-0.413 (-12.847)	-0.424 (-13.184)	-0.377 (-11.704)	-0.410 (-12.725)	-0.400 (-12.416)	-0.388 (-12.065)	-0.388 (-12.044)	-0.356 (-11.069)	-0.340 (-10.565)
<b>Excess Kurtosis</b>	1.728 (26.839)	1.619 (25.156)	1.395 (21.677)	1.521 (23.624)	1.400 (21.750)	1.335 (20.741)	1.501 (23.321)	1.284 (19.943)	1.216 (18.889)
<b>Jarque-Bera</b>	963.124	876.706	659.591	782.539	681.686	625.758	748.774	565.434	509.102

Table 6 reports the diagnostic tests of Models 2.1 to 2.9. Skewness and excess kurtosis are the estimated skewness and excess kurtosis of the standardized residuals from the mean equation. Model 2.1 is the benchmark model for Models 2.2 and 2.3. Model 2.4 is the benchmark model for Models 2.5 and 2.6. Model 2.7 is the benchmark model for Models 2.8 and 2.9. We estimate the models using daily close-to-close returns of S&P 500 index, implied variance, variance risk premium, and option market ambiguity attitudes over the period 2 Jan 1990 to 31 Dec 2014.

Table 7. Out-of-sample Forecasting

	Model 2.1	Model 2.2	Model 2.3	Model 2.4	Model 2.5	Model 2.6	Model 2.7	Model 2.8	Model 2.9
<b>LL</b>									
<i>RV</i> <sub>1min</sub>	1.2485	<b>1.1985</b> *** (0.008)	<b>1.2029</b> ** (0.011)	1.1260	<b>1.1170</b> (0.108)	<b>1.1218</b> (0.220)	1.1537	<b>1.1381</b> * (0.089)	<b>1.1274</b> * (0.062)
<i>RV</i> <sub>5min</sub>	1.3402	<b>1.2883</b> *** (0.006)	<b>1.2931</b> ** (0.016)	1.2147	<b>1.2049</b> * (0.097)	<b>1.2100</b> (0.213)	1.2438	<b>1.2281</b> * (0.095)	<b>1.2166</b> * (0.064)
<i>RV</i> <sub>10min</sub>	1.4120	<b>1.3588</b> *** (0.007)	<b>1.3634</b> *** (0.008)	1.2843	<b>1.2738</b> * (0.095)	<b>1.2791</b> (0.196)	1.3147	<b>1.2984</b> * (0.086)	<b>1.2865</b> ** (0.049)
<b>QLIKE</b>									
<i>RV</i> <sub>1min</sub>	0.4300	<b>0.4172</b> ** (0.031)	<b>0.4149</b> *** (0.008)	0.3908	<b>0.3874</b> * (0.064)	<b>0.3902</b> (0.264)	0.3990	<b>0.3942</b> * (0.097)	<b>0.3918</b> * (0.081)
<i>RV</i> <sub>5min</sub>	0.4427	<b>0.4279</b> ** (0.015)	<b>0.4286</b> ** (0.014)	0.4021	<b>0.3986</b> * (0.061)	<b>0.4008</b> (0.200)	0.4104	<b>0.4063</b> (0.117)	<b>0.4032</b> * (0.088)
<i>RV</i> <sub>10min</sub>	0.4725	<b>0.4588</b> ** (0.024)	<b>0.4568</b> *** (0.009)	0.4321	<b>0.4283</b> ** (0.047)	<b>0.4311</b> (0.236)	0.4413	<b>0.4366</b> * (0.093)	<b>0.4339</b> * (0.088)
<b>RMSE</b>									
<i>RV</i> <sub>1min</sub>	0.2721	<b>0.2555</b> * (0.072)	0.2893 (1.000)	0.3304	0.3360 (1.000)	0.3445 (1.000)	0.3908	<b>0.3742</b> ** (0.027)	<b>0.3861</b> (0.344)
<i>RV</i> <sub>5min</sub>	0.1859	<b>0.1599</b> * (0.083)	0.2121 (1.000)	0.2668	0.2736 (1.000)	0.2841 (1.000)	0.3393	<b>0.3201</b> ** (0.026)	<b>0.3345</b> (0.356)
<i>RV</i> <sub>10min</sub>	0.2590	<b>0.2410</b> * (0.084)	0.2795 (1.000)	0.3242	0.3297 (1.000)	0.3386 (1.000)	0.3868	<b>0.3701</b> ** (0.034)	<b>0.3833</b> (0.370)
<b>MAE</b>									
<i>RV</i> <sub>1min</sub>	0.0911	<b>0.0848</b> * (0.084)	0.0928 (1.000)	0.1029	0.1039 (1.000)	0.1052 (1.000)	0.1130	<b>0.1115</b> ** (0.032)	<b>0.1122</b> (0.353)
<i>RV</i> <sub>5min</sub>	0.0902	<b>0.0838</b> * (0.084)	0.0921 (1.000)	0.1023	0.1033 (1.000)	0.1047 (1.000)	0.1125	<b>0.1110</b> ** (0.036)	<b>0.1117</b> (0.369)
<i>RV</i> <sub>10min</sub>	0.0935	<b>0.0872</b> * (0.085)	0.0955 (1.000)	0.1059	0.1069 (1.000)	0.1082 (1.000)	0.1161	<b>0.1146</b> ** (0.029)	<b>0.1154</b> (0.356)

Table 7 reports error statistics from out-of-sample forecasting with 5-year rolling estimation windows. LL, QLIKE, RMSE, MAE are logarithmic loss function, loss implied by Gaussian likelihood, root mean squared error, and mean absolute error respectively. Loss functions are specified according to (21)-(24). Hansen (2005) SPA test p-values are reported in parentheses. Model 2.1 is the benchmark model for Models 2.2 and 2.3. Model 2.4 is the benchmark model for Models 2.5 and 2.6. Model 2.7 is the benchmark model for Models 2.8 and 2.9. Bolded figures are the errors that outperform the benchmark. We estimate the models using daily close-to-close returns of S&P 500 index, implied variance, variance risk premium, and option market ambiguity attitudes over the period 2 Jan 1990 to 31 Dec 2014. \*\*\*, \*\*, and \* indicate 99%, 95%, and 90% confidence levels respectively.

Table 8. Economic Significance Analysis

<i>Panel A. Economic Gain Based on Long-only Volatility-timing Trading Strategy</i>									
<i>FULL PERIOD</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
Cumulative Return	283.43%	<b>316.87%</b>	316.06%	269.16%	<b>327.93%</b>	298.49%	294.16%	<b>422.62%</b>	377.33%
Annualized Return	6.96%	7.41%	7.40%	6.76%	7.55%	7.17%	7.11%	8.63%	8.14%
Portfolio Volatility	18.51%	18.29%	18.96%	18.99%	18.97%	19.03%	19.01%	18.99%	19.08%
Return/risk Reward Ratio	0.38	0.40	0.39	0.36	0.40	0.38	0.37	0.45	0.43
Superiority to Benchmark Strategy (Percentage Points)		33.43	32.63		58.77	29.34		128.46	83.17
<i>1995 - 2004</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
Cumulative Return	151.20%	<b>178.49%</b>	170.67%	165.43%	178.84%	<b>186.64%</b>	157.76%	167.38%	<b>174.82%</b>
Annualized Return	9.68%	10.82%	10.50%	10.28%	10.83%	11.14%	9.96%	10.36%	10.67%
Portfolio Volatility	18.18%	17.98%	18.14%	18.16%	18.04%	18.09%	18.18%	18.15%	18.18%
Return/risk Reward Ratio	0.53	0.60	0.58	0.57	0.60	0.62	0.55	0.57	0.59
Superiority to Benchmark Strategy (Percentage Points)		27.29	19.47		13.41	21.20		9.62	17.06
<i>2005 - 2014</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
Cumulative Return	52.64%	49.69%	<b>53.72%</b>	39.08%	<b>53.47%</b>	39.02%	52.91%	<b>95.46%</b>	73.69%
Annualized Return	4.32%	4.12%	4.39%	3.35%	4.37%	3.35%	4.34%	6.93%	5.67%
Portfolio Volatility	18.84%	18.60%	19.74%	19.78%	19.86%	19.93%	19.81%	19.80%	19.94%
Return/risk Reward Ratio	0.23	0.22	0.22	0.17	0.22	0.17	0.22	0.35	0.28
Superiority to Benchmark Strategy (Percentage Points)		-2.95	1.08		14.39	-0.05		42.55	20.77
<i>Panel B. Economic Gain Based on Long/Short Volatility-timing Trading Strategy</i>									
<i>FULL PERIOD</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
Cumulative Return	227.33%	287.64%	<b>312.40%</b>	260.54%	<b>417.05%</b>	347.71%	296.45%	<b>610.31%</b>	509.87%
Annualized Return	6.12%	7.02%	7.35%	6.63%	8.57%	7.79%	7.14%	10.31%	9.47%
Portfolio Volatility	19.37%	19.37%	19.37%	19.37%	19.37%	19.38%	19.37%	19.36%	19.36%
Return/risk Reward Ratio	0.32	0.36	0.38	0.34	0.44	0.40	0.37	0.53	0.49
Superiority to Benchmark Strategy (Percentage Points)		60.31	85.07		156.51	87.17		313.86	213.42
<i>1995-2004</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
Cumulative Return	141.75%	<b>198.02%</b>	182.01%	172.43%	209.73%	<b>223.34%</b>	158.63%	180.79%	<b>196.44%</b>
Annualized Return	9.26%	11.57%	10.96%	10.57%	12.00%	12.49%	10.00%	10.91%	11.51%
Portfolio Volatility	18.25%	18.23%	18.24%	18.24%	18.24%	18.24%	18.24%	18.24%	18.23%
Return/risk Reward Ratio	0.51	0.63	0.60	0.58	0.66	0.68	0.55	0.60	0.63
Superiority to Benchmark Strategy (Percentage Points)		56.27	40.27		37.31	50.91		22.16	37.81
<i>2005-2014</i>	<i>Model 2.1</i>	<i>Model 2.2</i>	<i>Model 2.3</i>	<i>Model 2.4</i>	<i>Model 2.5</i>	<i>Model 2.6</i>	<i>Model 2.7</i>	<i>Model 2.8</i>	<i>Model 2.9</i>
Cumulative Return	35.40%	30.07%	<b>46.24%</b>	32.34%	<b>66.93%</b>	38.47%	53.29%	<b>152.96%</b>	105.73%
Annualized Return	3.08%	2.66%	3.87%	2.84%	5.26%	3.31%	4.36%	9.72%	7.48%
Portfolio Volatility	20.44%	20.45%	20.44%	20.44%	20.44%	20.45%	20.44%	20.42%	20.43%
Return/risk Reward Ratio	0.15	0.13	0.19	0.14	0.26	0.16	0.21	0.48	0.37
Superiority to Benchmark Strategy (Percentage Points)		-5.33	10.83		34.59	6.12		99.68	52.44

Table 8 reports the trading statistics for the long-only and long/short market-timing trading strategies based on variance forecasts from each model. Bolded figures are the cumulative returns of the best performing portfolios in each group. The trading window covers 2 Jan 1995 to 31 Dec 2014.

## TECHNICAL APPENDIX – Option Pricing under Ambiguity (Based on Driouchi et al., 2016)

Let  $B$  be the price of a riskless bond with instantaneous rate of return  $r$  such that:

$$\frac{dB}{B} = rdt \quad (A1)$$

Let  $O$  be the price of a contingent-claim (e.g., a European call or put option on the S&P index) which depends only on  $S$  and time  $t$ ,  $O(S, t)$ . From Ito's lemma and Eq. (1), the dynamics of option price  $O$  can be written ( $\forall m \in ]-1, 1[$ ,  $\forall s \in ]0, 1[$ ) as:

$$\begin{aligned} dO(S, t) = & \frac{\partial O}{\partial t} dt + \frac{\partial O}{\partial S} [(\mu + m\sigma)Sdt + (s\sigma)SdZ] + \frac{1}{2} \frac{d^2 O}{dS^2} [(s\sigma)SdZ]^2 \\ & + [(\mu + m\sigma)Sdt]^2 + [(s\sigma)SdZ \times (\mu + m\sigma)Sdt] \end{aligned} \quad (A2)$$

This simplifies to:

$$dO(S, t) = \left[ \frac{\partial O}{\partial t} + \frac{\partial O}{\partial S} (\mu + m\sigma)S + S^2 \frac{1}{2} \frac{d^2 O}{dS^2} (s\sigma)^2 \right] dt + \frac{\partial O}{\partial S} (s\sigma)SdZ \quad (A3)$$

Using Eq. (1), the level(s) of marginal utility in the economy  $\xi$  under Choquet ambiguity is:

$$\frac{d\xi}{\xi} = [mg(\xi, S) + f(\xi, S)]dt + sg(\xi, S)dz \quad (A4)$$

This results from standard economic dynamics  $\frac{d\xi}{\xi} = f(\xi, S)dt + g(\xi, S)dW$  (see Harrison and Kreps, 1979)

and the characteristics of  $W$  in the Choquet ambiguity universe. Functions  $g$  and  $f$  help derive the pricing kernel under uncertainty. Thus:

$$\begin{aligned} d(\xi B) &= \xi(rBdt) + B[(mg(\xi, S) + f(\xi, S))\xi dt + sg(\xi, S)\xi dz] \\ &= \xi B[(r + mg(\xi, S) + f(\xi, S))dt + sg(\xi, S)\xi dz] \end{aligned} \quad (A5)$$

Applying martingale theory, the drift (dt) term is set to zero. This implies:

$$r + mg(\xi, S) + f(\xi, S) = 0 \text{ or } f(\xi, S) = -r - mg(\xi, S) \quad (A6)$$

Following a similar procedure for  $S$ :

$$d(\xi S) = \xi dS + S d\xi + d \langle \xi, S \rangle \quad (A7)$$

$$\begin{aligned} d(\xi S) &= \xi S[(\mu + m\sigma)dt + s\sigma dz] + S\xi\{[mg(\xi, S) - r - mg(\xi, S)]dt + sg(\xi, S)dz\} + s^2\sigma\xi Sg(\xi, S)dt \\ &= \xi S[(\mu + m\sigma) - r + s^2\sigma g(\xi, S)]dt + S\xi[s\sigma + sg(\xi, S)]dz \end{aligned} \quad (A8)$$

Setting the drift term to zero, we obtain the ambiguity-adjusted Sharpe ratio  $g(\xi, S)$ :

$$(\mu + m\sigma) - r + s^2\sigma g(\xi, S) = 0 \text{ or } g(\xi, S) = \frac{[r - (\mu + m\sigma)]}{s^2\sigma} \quad (A9)$$

The market pricing kernel follows Harrison and Kreps (1979) dynamics but, due to market incompleteness, multiple marginal utility levels and Knightian uncertainty,  $f$  and  $g$  are not unique as they are affected by investors' ambiguity parameters  $m$  and  $s$ . This means that Choquet ambiguity impacts the fundamental component of the market pricing kernel (via parameters  $m$  and  $s$ ) but not the purely sentimental element (see Cochrane, 2001; Shefrin, 2005). Relaxing this general (market incompleteness) assumption reduces to the perfect replication or risk-neutral case of Black-Scholes (1973) OPM. Using the results from Eqs. (A6) and (A9):

$$\begin{aligned} \frac{d\xi}{\xi} &= f(\xi, S)dt + g(\xi, S)dz \\ &= -r - m\left\{\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right\}dt + \left(\frac{[r - (\mu + m\sigma)]}{s^2\sigma}\right)dz \end{aligned} \quad (A10)$$

Consider the value of a call or put option  $O$  written on underlying stock index  $S$  (with dividend yield  $\delta$ ).

$$\begin{aligned}
d(\xi O) &= \xi dO + O d\xi + d < \xi, O > \\
&= \xi \left\{ \left[ \frac{\partial O}{\partial t} + \frac{\partial O}{\partial S} (\mu - \delta + m\sigma)S + S^2 \frac{1}{2} \frac{d^2 O}{dS^2} (s\sigma)^2 \right] dt + \frac{\partial O}{\partial S} (s\sigma) S dz \right\} \\
&+ \xi O \left[ -r - m \left\{ \frac{[r - (\mu + m\sigma)]}{s^2 \sigma} \right\} dt + \left( \frac{[r - (\mu + m\sigma)]}{s^2 \sigma} \right) dz \right] \\
&+ \xi \left[ \left( \frac{[r - (\mu + m\sigma)]}{s^2 \sigma} \right) \frac{\partial O}{\partial S} (s\sigma) S dt \right] \tag{A11}
\end{aligned}$$

Setting the drift (dt) term of the option to zero results in the fundamental equation for pricing derivatives or contingent-claims  $O_{c/p}$ :

$$\begin{aligned}
&\xi \left[ \frac{\partial O}{\partial t} + \frac{\partial O}{\partial S} (\mu - \delta + m\sigma)S + S^2 \frac{1}{2} \frac{d^2 O}{dS^2} (s\sigma)^2 \right] dt + \xi O \left[ -r - m \left\{ \frac{[r - (\mu + m\sigma)]}{s^2 \sigma} \right\} dt \right] \\
&+ \xi \left[ \left( \frac{[r - (\mu + m\sigma)]}{s^2 \sigma} \right) \frac{\partial O}{\partial S} (s\sigma) S dt \right] = 0 \tag{A12}
\end{aligned}$$

Solving Eq. (A12) for European options written on  $S$  leads to Eqs. (3-5).

### References for the Technical Appendix:

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